NOTES

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The Mathematical Relationship Between Kepler's Laws and Newton's Laws

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1. INTRODUCTION. Whenever a new scientific theory comes down the pike, it is greeted by skeptics who demand proof that the new theory is as good as the theory it would displace. That is why "the major scientific problem of the [seventeenth] century" was to prove that Isaac Newton's law of gravity gives the same correct results as the older laws of Johannes Kepler [**4**]. This famous mathematical problem is solved below in an innovative way that requires no trigonometry, only elementary calculus, and none of the usual "clever tricks" [**8**].

Supposing that planets move according to Kepler's Laws (which are reviewed in Section 2 below), then it follows that planetary acceleration is given by Newton's central inverse-square equation (which is equation twelve below). This historic theorem was first proved by Newton, who thereby established his law of gravity as a respectable successor to Kepler's Laws. This same theorem is proved in Section 3, using simple and straightforward methods. The reverse theorem, according to which the central $1/R^2$ equation requires Keplerian orbits, is proved in Section 4.

The two theorems proved here were first published in Newton's 1687 *Philosophiae Naturalis Principia Mathematica*, or *Principia* for short. Newton admitted that the *Principia* is purposely "abstruse" ([**3**], p. 90), and a controversy persists as to whether Newton's proofs are entirely legitimate ([**2**], p. 30). Unlike the *Principia*, the brief proofs below are quite transparent.

Kepler's Laws are differentiated in Section 3 using only Cartesian coordinates, and this novel Cartesian approach contrasts with the usual technique of transforming to polar coordinates. Although the converse proof of Section 4 is fundamentally the same as those of a few other authors ([**5**], p. 178 of [**11**], and p. 625 of [**1**]), each step in Section 4 follows naturally and inexorably from what precedes it. No rabbits are pulled out of hats. The method of Section 4 is thus presented in a clear manner which compares favorably to the more common methods of solving the same problem, and also to various uncommon methods which are discussed in [**10**].

2. REVIEW OF KEPLER'S LAWS. Kepler deduced his laws from data supplied by the astronomer Tycho Brahe. Kepler's Laws are:

I. *Each planet moves along an ellipse with the Sun at a focus.*
II. *The line from a planet to the Sun sweeps out equal areas in equal times.*
III. *The square of a revolution's duration, divided by the cube of the orbit's greatest width, is the same for all planets.*

Kepler introduced the first two laws in his 1609 *Astronomia Nova*. The third or "harmonic" law was suggested in his 1619 *Harmonice Mundi*, and is often stated in terms of the length "a" of the semimajor axis ("a" is half the orbit's greatest width). The discovery of these laws marked the greatest advance since Aristarchus deduced nineteen centuries earlier that planets circle the Sun (see p. 2 of [**6**]).

Ellipses are, of course, the closed curves formed by intersecting a cone and a plane. They were studied by the ancient Greeks (see p. 119 of [**6**]) who proved that the distance to a point (the "focus") divided by the distance to a line (the "directrix") is a constant "eccentricity" ε. A beautiful proof of this focus-directrix property was devised in 1822 by G. P. Dandelin. Dandelin's proof appears at p. 546 of [**9**], and it applies to both closed ($0 \leq \varepsilon < 1$) and open ($\varepsilon \geq 1$) conic sections.

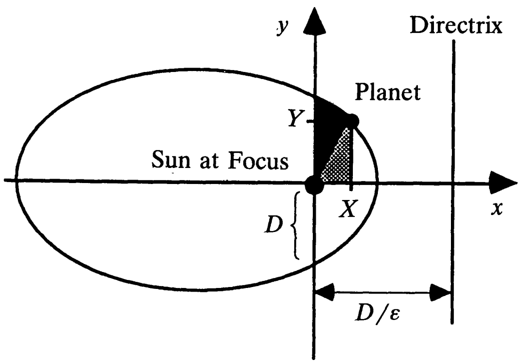

Figure 1

Kepler's Laws can be translated into equations by picturing a planet as a point-particle in the x-y plane, having coordinates (X, Y) at time t (see Figure). The Sun is located at the origin, and the planet's directrix is perpendicular to the x-axis at a distance D/ε from the Sun. "D" is called the "semi-latus-rectum" of the conic section. According to Kepler's First Law, the distance $R \equiv \sqrt{X^2 + Y^2}$ from the planet to the Sun is given by:

$$R = D - \varepsilon X. \tag{1}$$

Kepler's Second Law can be formulated in similarly simple terms. If the planet crosses the y-axis at time t_0, then the area swept between t_0 and t equals the area under the curve minus the triangular area beneath the line from Sun to planet. Hence, at all times,

$$\int_0^X y\,dx - XY/2 = C(t - t_0) \tag{2}$$

where "C" is the constant ratio of area swept to time elapsed (a new constant t_0 must be introduced whenever the planet crosses the x-axis).

The orbit's total area divided by a revolution's duration is clearly equal to C. Also, it is not difficult to prove that the area of an ellipse is πab with $a = D[1 - \varepsilon^2]^{-1}$ and $b = D[1 - \varepsilon^2]^{-1/2}$ (these two equations can be easily derived using equation one). Therefore, Kepler's Third Law is:

$$C^2/D = K \tag{3}$$

where the constant "K" is the same for all planets. In summary, Kepler's Laws are (1), (2), and (3).

3. PROOF OF CENTRAL $1/R^2$ EQUATION. Kepler's Laws will now be used to find the acceleration of a planet. Differentiating (1) produces:

$$\frac{1}{R}\left[X\frac{dX}{dt} + Y\frac{dY}{dt}\right] = -\varepsilon\frac{dX}{dt}. \tag{4}$$

Differentiating (2), using the Fundamental Theorem of Calculus, gives:

$$Y\frac{dX}{dt} - X\frac{dY}{dt} = 2C. \tag{5}$$

A bit of algebra applied to (4), (5), and (1) makes it clear that the two velocity components are:

$$\frac{dX}{dt} = \frac{2C}{D}\cdot\frac{Y}{R} \tag{6}$$

and

$$\frac{dY}{dt} = -\frac{2C}{D}\cdot\frac{X}{R} - \frac{2C\varepsilon}{D}. \tag{7}$$

Differentiating (5) yields:

$$Y\frac{d^2X}{dt^2} - X\frac{d^2Y}{dt^2} = 0. \tag{8}$$

Differentiation of the right-hand-side of (6) is facilitated by the following identity:

$$\frac{d}{dt}\left[\frac{Y}{R}\right] = \frac{X}{R^3}\cdot\left[X\frac{dY}{dt} - Y\frac{dX}{dt}\right]. \tag{9}$$

This identity is based solely upon the definition of R.[†] By differentiating (6) and plugging in (9), (5) and (3) one gets:

$$\frac{d^2X}{dt^2} = \frac{-4KX}{R^3}. \tag{10}$$

By (8) and (10),

$$\frac{d^2Y}{dt^2} = \frac{-4KY}{R^3}. \tag{11}$$

Equations (10) and (11) can be written compactly in terms of vectors.

$$\frac{d^2\vec{R}}{dt^2} = \frac{-4K\vec{R}}{R^3}. \tag{12}$$

Equation (12) is Newton's central inverse-square equation. This equation expresses Newton's law of gravity for the special case where planetary mass is negligible.

[†]Incidentally, note that $[X\dot{Y} - Y\dot{X}]$ is twice the areal speed (i.e., $R^2\dot{\theta}$ in polar coordinates), where dots denote differentiation. The referee has keenly observed that therefore equation (9) is basically $[\sin\theta]^{\cdot} = [\cos\theta]\dot{\theta}$.

4. RECOVERY OF KEPLER'S LAWS. It remains to be seen whether a bounded orbit could satisfy (12) if it is not Keplerian. In other words, could a planet be accelerating according to (12), and yet violate Kepler's Law? It will now be proved that such an orbit is impossible, by recovering Kepler's Laws from (12). By the way, it is taken for granted that motion is confined to a plane, though this assumption is easily justified ([**7**], p. 105).

Equations (10) and (11) lead to (8), and integrating (8) retrieves (5) and (2). Plugging (5) into the crucial identity (9) gives:

$$\frac{d}{dt}\left[\frac{Y}{R}\right] = \frac{-2CX}{R^3}. \tag{13}$$

On account of (13) and (10),

$$\frac{Y}{R} = \frac{C}{2K} \cdot \frac{dX}{dt} + A \tag{14}$$

where "A" is a constant of integration.

The identity (9) has been very useful here, and it would have been necessary to pull this identity out of thin air were it not for the context provided by Section 3. In this context, the identity (9) has arisen in a natural way (whereas other authors have indeed pulled this identity from out of the blue).

Interchanging "X" and "Y" in (9) produces another identity which together with (5) yields:

$$\frac{d}{dt}\left[\frac{X}{R}\right] = \frac{2CY}{R^3}. \tag{15}$$

So, by (11),

$$\frac{X}{R} = \frac{-C}{2K} \cdot \frac{dY}{dt} + B \tag{16}$$

where "B" is another constant. Plugging (14) and (16) into (5) yields:

$$\frac{C^2}{K} + AY + BX = R. \tag{17}$$

If $A = B = 0$, this describes a circle. If not, (17) represents a conic section with focus at the origin, eccentricity $[A^2 + B^2]^{1/2}$, and directrix given by:

$$\frac{C^2}{K} + Ay + Bx = 0. \tag{18}$$

This interpretation of (17) follows from a simple fact of analytic geometry: the distance from a point (x_0, y_0) to a line $ax + by + c = 0$ is equal to $|ax_0 + by_0 + c|[a^2 + b^2]^{-1/2}$. This well-known fact can also be applied to (18) in order to find the distance from focus to directrix, and it is thus evident that the focus-directrix distance is as described by (3). Consequently, if Newton's central inverse-square equation holds true then all bounded orbits must satisfy Kepler's Laws, which was to be demonstrated.

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11. A. Wintner, *The Analytical Foundations of Celestial Mechanics*, Princeton Univ. Press, 1941. I thank Dr. Robert Weinstock of Oberlin College for telling me of this book, and for his many helpful comments. Dr. Weinstock notes that Wintner's own references are incorrect (see ref. 20 of **[10]**).

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A Short Proof of a Result on Polynomials

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In this note we want to present a short proof of a result that appeared in **[1]**. For a polynomial $f(x) = \prod_1^n(x - x_i)$, with distinct real roots $x_1 < x_2 < \cdots < x_n$, we let $d = \delta(f) = \min_i(x_{i+1} - x_i)$ and $g(x) = f'(x)/f(x) = \sum_1^n 1/(x - x_i)$. If k is a real number then the roots of the polynomial $f' - kf$ are also real and distinct.

Proposition. *If for some j, y_0 and y_1 satisfy $y_0 < x_j < y_1 \le y_0 + d$ then y_0 and y_1 are not zeros of f and $g(y_0) < g(y_1)$.*

Proof: The hypothesis implies that for all i, $y_1 - y_0 \le d \le x_{i+1} - x_i$. Hence for $1 \le i \le j - 1$ we have $y_0 - x_i \ge y_1 - x_{i+1} > 0$ and so $1/(y_0 - x_i) \le 1/(y_1 - x_{i+1})$; similarly for $j \le i \le n - 1$ we have $y_1 - x_{i+1} \le y_0 - x_i < 0$ and again $1/(y_0 - x_i) \le 1/(y_1 - x_{i+1})$.

Finally $y_0 - x_n < 0 < y_1 - x_1$, so $1/(y_0 - x_n) < 0 < 1/(y_1 - x_1)$, and the result follows by addition of these inequalities.

Corollary. $\delta(f' - kf) > \delta(f)$.